

Yang-Mills Solutions and Dyons on Cylinders over Coset Spaces with Sasakian Structure

Maike Tormählen

*Institut für Theoretische Physik,
Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*

Abstract</

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2 Sasakian manifolds

As described in the introduction, compact G -structure manifolds play a key role in string compactifications. Particularly interesting in this context are real Killing spinor manifolds (see also [23] for details). Besides the round spheres, these are

- 6-dimensional $SU(3)$ -structure manifolds,
- 7-dimensional G_2 -structure manifolds,
- $(2m+1)$ -dimensional Sasakian manifolds with structure group $SU(m)$,
- $(4m+3)$ -dimensional 3-Sasakian manifolds with structure group $Sp(m)$.

The torsion term in the Yang-Mills equation is generated by variation of the Chern-Simons term, while the other summands arise from variation of the Yang-Mills term.

For a motivation of the following discussion, recall that a connection with totally antisymmetric torsion naturally appears in the conditions for supersymmetry preservation in heterotic supergravity [29]. On suitably chosen string backgrounds, one can introduce geometric three-form fluxes that are identified with the torsion of this spin connection. The Yang-Mills equation follows from covariant differentiation of the higher-dimensional instanton equation if the three-form is related to the G -structure four-form as $*\mathcal{H} := d*Q$. In this case, the Yang-Mills equation is the equation of motion of the action (3.3).

Non-BPS Yang-Mills solutions can be constructed when the Yang-Mills equation is not required to follow from a first-order equation. In accordance with earlier work [7, 8, 22], we choose to identify the three-form \mathcal{H} with the torsion of the spin connection, $\mathcal{H}_{ABC} \propto T_{ABC}$, and the torsion components⁴ to be proportional to the structure constants on G/H :

$$T_{abc} = \kappa f_{abc}, \quad \kappa \in \mathbb{R}. \quad (3.4)$$

In explicit examples, the relation of T and \mathcal{H} will be chosen such that $*\mathcal{H} = d*Q$ is satisfied for $\kappa = 1$ and the Yang-Mills equation follows from the instanton equation for this value of κ . Other choices are possible and correspond to a rescaling of the parameter κ . Solutions of the torsionful Yang-Mills equation can be lifted to solutions of heterotic supergravity if they follow from a first-order BPS equation. More general non-BPS Yang-Mills solutions for arbitrary values of κ can potentially serve as building blocks for non-supersymmetric string solutions.

Written out in components, the torsionful Yang-Mills equation on the product space $\mathbb{R} \times G/H$ turns into the following set of equations, where the metric g is assumed to be of diagonal form with coordinate-dependent components:

$$\begin{aligned} & \frac{g_{BB}}{\sqrt{|g|}} \partial_C \left(\sqrt{|g|} \mathcal{F}^{CB} \right) \\ & - \mathcal{F}^{CD} \left(\frac{1}{2} T_{CDB} - \Gamma_{CDB} \right) + \mathcal{F}^C{}_B \left(\frac{1}{2} T_{CD}{}^D - \Gamma_{CD}{}^D \right) \\ & - \mathcal{F}^C{}_B \left(\frac{1}{2} T_{DC}{}^D - \Gamma_{DC}{}^D \right) + [\mathcal{A}^A, F_{AB}] - \frac{1}{2} \mathcal{H}_{CDB} \mathcal{F}^{CD} = 0. \end{aligned} \quad (3.5)$$

We study the Yang-Mills equation on the cylinder $\mathcal{Z}(M) = \mathbb{R} \times M$ with metric (2.19), where $M = G/H$ is a coset space with Sasakian structure of dimension $2m + 1$. In this setup, G is a semisimple Lie group and H a closed Lie subgroup. Capital indices $A = \{0, 1, 2, \dots, 2m + 1\}$ are used to label all $2m + 2$ directions of the product manifold. The upper index will always be pulled down *behind* the lower two ones. This convention is important, as not all quantities do a priori have totally antisymmetric indices.

⁴It has been argued in [8] that for such a choice of \mathcal{H}_{ABC} and T_{ABC} , the Yang-Mills equation on the cylinder over a nearly-Kähler coset space follows from an action similar to (3.3). This does not have to hold for other choices of \mathcal{H} .

As the free index B runs from 0 to $\dim G/H$, these are $2m + 2$ equations. The coefficients \mathcal{H}_{ABC} (with all indices lowered) are the components of the 3-form \mathcal{H} , and $-\Gamma_{AB}^C$ are the coefficients of the torsionful spin connection with torsion T_{AB}^C . Equation (3.5)⁵ has been discussed in detail on the cylinder over an arbitrary coset space G/H with gauge connection $\mathcal{A} = e^i I_i + \phi e^a I_a$ in [7, 8, 21, 22], leading to explicit kink-type solutions.

For further specification, we have to compute the components of the 3-form \mathcal{H} . According to [20], an invariant 4-form on the cylinder over a Sasakian manifold can be constructed as

$$Q = \frac{2m}{m+1} d\tau \wedge P_M + \left(\frac{2m}{m+1} \right)^2 Q_M. \quad (3.6)$$

A direct computation yields

$$* d * Q = -\frac{5}{2 \cdot 3!} f^{KL} {}_{[R} Q_{KLPQ]} e^{PQR}. \quad (3.7)$$

Using the definition $*\mathcal{H} := d*Q$ and decomposition rules for antisymmetric tensor indices, we find

$$\mathcal{H}_{PQR} = -\frac{5}{2} f^{KL} {}_{[R} Q_{KLPQ]} = -\frac{15}{2} Q_{KL[PQ} f^{KL}{}_{R]}. \quad (3.8)$$

At this point, we have to distinguish between indices in cylinder direction (0), contact direction (1) and all other directions and find that the following components vanish for all m :

$$\mathcal{H}_{01r} = \mathcal{H}_{0qr} = \mathcal{H}_{pqr} = 0. \quad (3.9)$$

The remaining components depend on the value of m . We demonstrate this by writing out \mathcal{H}_{231} explicitly, using equations (2.5), (2.14) and (2.22). All other nonvanishing components of \mathcal{H} behave in a similar way:

$$\begin{aligned} \mathcal{H}_{231} &= -2P_{231}Q_{2323} = 0 && \text{for } m = 1, \\ \mathcal{H}_{231} &= -\frac{9}{16}P_{mn1}Q_{mn23} = -2P_{451} = -f_{231} && \text{for } m = 2, \\ \mathcal{H}_{231} &= -\frac{4}{9}P_{mn1}Q_{mn23} = -2(P_{451} + P_{671}) = -2f_{231} && \text{for } m = 3, \\ \mathcal{H}_{231} &= (1-m)f_{231} && \text{for arbitrary } m. \end{aligned} \quad (3.10)$$

Note that the case $m = 1$ of lowest dimension with $\mathcal{H} = 0$ is special. We will not further discuss it here. In order to recover the instanton case for $\kappa = 1$, we choose

$$H_{\mu\nu\rho} = (1-m)T_{\mu\nu\rho} = (1-m)\kappa f_{\mu\nu\rho}. \quad (3.11)$$

⁵Note that this equation is not identical to the corresponding equations (2.19) and (2.20) in [22] due to differently normalized torsion. The equations presented in the reference follow from our equation (3.5) with cylinder metric $g_{\mathcal{Z}} = d\tau^2 + \delta_{ab}e^a e^b$ in the special case of $H_{ABC} = -T_{ABC}$.

With this choice and the cylinder metric (2.19), equation (3.5) turns into

$$\begin{aligned} \partial_A \mathcal{F}^{AB} - \mathcal{F}^{CD} \left(\frac{1}{2}(2-m)T_{CD}^B - \Gamma_{CD}^B \right) \\ + \mathcal{F}^{CB} \left(\frac{1}{2}T_{CD}^D - \Gamma_{CD}^D \right) - \mathcal{F}^{CB} \left(\frac{1}{2}T_{DC}^D - \Gamma_{DC}^D \right) + [\mathcal{A}_A, \mathcal{F}^{AB}] = 0, \end{aligned} \quad (3.12)$$

where Γ denotes the torsionful spin connection. The $B = 0$ equation is identically satisfied. Let us take a look at the cases with $B > 0$. The summand $\mathcal{F}^{C\mu}(\frac{1}{2}T_{CD}^D - \Gamma_{CD}^D)$ vanishes identically. From the summand $\mathcal{F}^{C\mu}(\frac{1}{2}T_{DC}^D - \Gamma_{DC}^D)$, as well as from $[\mathcal{A}_A, \mathcal{F}^{AB}]$, we obtain terms proportional to the functions e_μ^i . These terms add up to zero by use of the Jacobi identity and $SU(m)$ -equivariance of the connection and will therefore be omitted in the following computation. We evaluate the remaining terms explicitly. The connection coefficients are derived from the Maurer-Cartan structure equation

$$T^A = de^A + \Gamma_{BC}^A e^{BC}. \quad (3.13)$$

With

$$de^A = -\frac{1}{2}f_{BC}^A e^{BC} \quad \text{and} \quad T_{BC}^A = \kappa f_{BC}^A, \quad (3.14)$$

where $\kappa \in \mathbb{R}$ is a real parameter, they take the form

$$\Gamma_{bc}^1 = \frac{1}{2}(\kappa + 1)f_{bc}^1, \quad (3.15)$$

$$\Gamma_{1b}^a = \frac{1}{2}(\kappa + 1)f_{1b}^a + f_{ib}^a e_c^i, \quad (3.16)$$

$$\Gamma_{bc}^a = f_{ic}^a e_b^i. \quad (3.17)$$

By construction, the value $\kappa = 1$ describes the instanton case discussed in [18–20]. For the derivation of explicit second-order equations, we will use

$$T^1 = P_{1ab}e^{1ab}, \quad T^a = \frac{m+1}{2m}P_{a\mu\nu}e^{a\mu\nu}, \quad (3.18)$$

along with equations (2.22). Using equation (2.20) and omitting the τ -dependence of the functions χ and ψ , we obtain the following curvature:

$$\begin{aligned} \mathcal{F} = -\frac{1}{2} \left(1 - \frac{1}{2m}\psi^2 \right) f_{ab}^i e^{ab} I_i + \dot{\chi} e^{01} I_1 + \frac{1}{\sqrt{2m}} \dot{\psi} e^{0a} I_a \\ + \left(\chi - \frac{1}{2m}\psi^2 \right) P_{ab1} e^{ab} I_1 + \frac{m+1}{m\sqrt{2m}} \psi (1 - \chi) P_{1ba} I_a e^{1b}. \end{aligned} \quad (3.19)$$

Inserting \mathcal{F} , T^A , ω_{AB}^C as above and using

$$f_{ac}^i f_{ib}^c = \frac{2(m^2 - 1)}{m} \delta_{ab}, \quad (3.20)$$

equation (3.12) turns into

$$\ddot{\chi} = \frac{(m+1)^2}{m} \left(((m-1)\kappa + 1)\chi - ((m-1)\kappa + 3) \frac{1}{2m}\psi^2 + \frac{1}{m}\chi\psi^2 \right), \quad (3.21a)$$

$$\ddot{\psi} = \left(\frac{m+1}{m} \right)^2 \psi \left((m-1)\kappa + 2 - m - ((m-1)\kappa + 3)\chi + \chi^2 + \frac{1}{2}\psi^2 \right). \quad (3.21b)$$

The derivation of the identity (3.20) can be found in Appendix A.

4 Action functional and potential

The second-order equations (3.21) are equations of motion for the action

$$\begin{aligned}
S &= \frac{m}{4(m+1)} \int_{\mathbb{R} \times M} \text{tr} \left(\mathcal{F} \wedge * \mathcal{F} + 2 \left(\frac{m}{m+1} \right)^2 \kappa d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F} \right) \\
&= \text{Vol}(M) \times \int_{\mathbb{R}} \left[-\frac{1}{2} (\dot{\chi}^2 + \dot{\psi}^2) - \left(\frac{m+1}{m} \right)^2 \right. \\
&\quad \left(\psi^2 (1-\chi)^2 + m(1-m)(1-\kappa) \left(\frac{1}{2m} \psi^2 - 1 \right)^2 \right. \\
&\quad \left. \left. + m(1+\kappa(m-1)) \left(\chi - \frac{1}{2m} \psi^2 \right)^2 \right) \right] d\tau \quad (4.1)
\end{aligned}$$

with potential

$$\begin{aligned}
V(\chi, \psi) &= \frac{1}{2} \left(\frac{m+1}{m} \right)^2 \left(((1+\kappa(m-1))m\chi^2 + (\kappa(1-m)-3)\chi\psi^2 \right. \\
&\quad \left. + \chi^2\psi^2 + (2-m+\kappa(m-1))\psi^2 + \frac{1}{4}\psi^4 + m(m-1)(1-\kappa) \right), \quad (4.2)
\end{aligned}$$

where $*_M$ denotes the Hodge star operator on the Sasakian manifold M with respect to the metric $g_M = e^1 e^1 + \frac{2m}{m+1} \delta_{ab} e^a e^b$, $*$ denotes the Hodge star operator on the cylinder, and $\text{Vol}(M) = \sqrt{|g_M|} e^{1,2,\dots,2m+1}$ is the volume form on M . This can be verified by a direct computation, presented in Appendix B. Equations (3.21) constitute a gradient system of the form

$$\begin{pmatrix} \ddot{\chi} \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} \partial_\chi \\ \partial_\psi \end{pmatrix} V. \quad (4.3)$$

With our sign convention, this model describes a particle moving in the potential $-V$. The potential V is symmetric with respect to sign changes of ψ and has the following critical points (i.e. $\ddot{\chi} = \ddot{\psi} = 0$) for arbitrary m, κ :

$$\begin{aligned}
(\chi_1, \psi_1) &= (0, 0), \\
(\chi_2, \psi_2) &= (1, \pm \sqrt{2m}), \\
(\chi_3, \psi_3) &= \left(\frac{1}{4} \left(7 + 3(m-1)\kappa + \sqrt{P} \right), \right. \\
&\quad \left. \pm \frac{1}{2} \sqrt{((1-m)\kappa - 1) \left((1-m)\kappa - 1 + 4m + \sqrt{P} \right)} \right), \\
(\chi_4, \psi_4) &= \left(\frac{1}{4} \left(7 + 3(m-1)\kappa + \sqrt{P} \right), \right. \\
&\quad \left. \pm \frac{1}{2} \sqrt{((1-m)\kappa - 1) \left((1-m)\kappa - 1 + 4m - \sqrt{P} \right)} \right), \quad (4.4)
\end{aligned}$$

	κ	Eigenvalues of Jacobian
$(\chi_1, \psi_1) = (0, 0)$	1	$(m+1)^2, \frac{(m+1)^2}{2}$
$(\chi_2, \psi_2) = (1, \pm\sqrt{2m})$	any	see Appendix C
$(\chi_3, \psi_3) = (1, -\sqrt{2m})$	$\frac{m-2-\sqrt{m(8+m)}}{2(m-1)}$	0, positive
$(\chi_4, \psi_4) = (-1, \pm\sqrt{2m})$	$\frac{3}{1-m}$	$\frac{(m+1)^2(m-\sqrt{m(m+8)})}{m^2}, \frac{(m+1)^2(m+\sqrt{m(m+8)})}{m^2}$
$(1, \sqrt{2m})$	$\frac{m-2+\sqrt{m(8+m)}}{2(m-1)}$	0, positive

Table 1: Critical points and corresponding κ values with vanishing potential

where the abbreviation

$$P = (m-1)^2\kappa^2 + \kappa(8m^2 - 6m - 2) + 24m + 1 \quad (4.5)$$

is used. Finite-action Yang-Mills solutions $\chi(\tau), \psi(\tau)$ must interpolate between zero potential critical points. With κ arbitrary, the potential vanishes for the second critical point $(\chi_2, \psi_2) = (1, \pm\sqrt{2m})$. We find $V(\chi_1, \psi_1) = \frac{(\kappa-1)(m-1)(m+1)^2}{2m}$ for the first critical point, which vanishes only for $\kappa = 1$, as well as lengthy nonzero expressions for $V(\chi_3, \psi_3)$ and $V(\chi_4, \psi_4)$. The critical points are listed in Table 1, together with the κ -values for which their potential becomes zero. For the values of κ listed in Table 2, more than two critical points are located on the same axis, and hence the system may admit analytic solutions. In addition, we note that at $\kappa = \frac{m-2}{m-1}$, five of the seven critical points coincide at $(0, 0)$, at $\kappa = \frac{m-2-\sqrt{m(8+m)}}{2(m-1)}$ the point (χ_3, ψ_3) coincides with (χ_2, ψ_2) and (χ_4, ψ_4) becomes imaginary, and at $\kappa = \frac{2-m-\sqrt{m(8+m)}}{2(m-1)}$, (χ_4, ψ_4) coincides with (χ_2, ψ_2) and (χ_3, ψ_3) becomes imaginary.

4.1 Analytic Yang-Mills solutions

Equations (3.21) constitute a system of nonlinear coupled differential equations, hence we cannot expect to be able to find analytic solutions. The case $\kappa = \frac{1}{1-m}$, however, admits an analytic solution to the Yang-Mills equation, interpolating between the critical points $(1, \sqrt{2m})$ and $(1, -\sqrt{2m})$ for arbitrary m . All other critical points are located on the

κ	Critical points	$V(\text{critical points})$
$\frac{1}{1-m}$	$(0, 0), (1, \pm\sqrt{2m}), (1 \pm m, 0)$	$\frac{1}{2}(m+1)^2, 0, \frac{1}{2}(m+1)^2$
$\frac{3}{1-m}$	$(0, 0), (1, \pm\sqrt{2m}), (-1, \pm\sqrt{2m}),$ $(0, \pm\sqrt{2(m+1)})$	$\frac{(m+1)^2(m+2)}{2m}, 0, 0, -\frac{(m+1)^2}{2m^2}$

Table 2: Values of κ for which more than two critical points lie on the same axis

χ -axis and have potential $V = \frac{1}{2}(m+1)^2$. The zero-potential critical points are therefore minima of V , and we expect to find interpolating finite-action Yang-Mills solutions. With $\chi = 1$, equations (3.21) take the form

$$\ddot{\chi} = 0, \tag{4.6a}$$

$$\ddot{\psi} = \frac{(m+1)^2}{m}\psi \left(\frac{1}{2m}\psi^2 - 1 \right). \tag{4.6b}$$

Equation (4.6) is solved by

$$\psi = \pm\sqrt{2m} \tanh \left(\pm \frac{m+1}{\sqrt{2m}}\tau \right). \tag{4.7}$$

This is a kink solution with finite energy and finite action. A plot of this solution in the (χ, ψ) -plane can be found in Figure 3.

For $\kappa = \frac{3}{1-m}$, there are three critical points on the $\chi = 0$ axis. However, none of them has zero potential, and we do not find any analytic solutions.

4.2 Periodic solutions

A different kind of solutions is obtained by changing from $\mathbb{R} \times M$ to $S^1 \times M$, i. e. when the additional direction is not a real line but a unit circle with circumference L . In this case, periodic boundary conditions have to be imposed:

$$\psi(\tau) = \psi(\tau + L). \tag{4.8}$$

We restrict the consideration to the analytically solvable case (4.6), which has the periodic solution

$$\psi(\tau) = \pm \frac{2k\sqrt{m}}{\sqrt{1+k^2}} \operatorname{sn} \left[\frac{m+1}{\sqrt{m(1+k^2)}}\tau; k \right]. \tag{4.9}$$

This solution is known as a sphaleron [30]. Sphalerons are unstable solutions of the classical equations of motion. $\text{sn}[u, k]$ with $0 \leq k \leq 1$ is a Jacobi elliptic function, details of which can be found for example in Appendix B of [22] or in [31]. The Jacobi elliptic function has a period of $4K(k)$, where $K(k)$ denotes the complete elliptic integral of the first kind. The boundary condition (4.9) therefore turns into

$$4K(k)n = \frac{m+1}{\sqrt{m(1+k^2)}}L, \quad n \in \mathbb{N}, \quad (4.10)$$

fixing $k = k(L, n)$ and $\psi(\tau; k(L, n)) =: \psi^{(n)}(\tau)$. Solutions (4.9) exist if $L \geq 2^{\frac{3}{2}}\pi n$ (cf. [21, 22]). The topological charge of the sphaleron $\psi(n)$ is zero due to the periodic boundary conditions. This solution is interpreted as a configuration of n kinks and n antikinks, alternating and equally spaced around the circle. The tanh-solution from Chapter 4.1 arises from the Jacobi elliptic function in the limit $k \rightarrow 1$. In the limit $k \rightarrow 0$, the elliptic function approaches $\sin\left(\frac{m+1}{\sqrt{m(1+k^2)}}\tau\right)$. In analogy to results in [32], our solution (4.9) with positive sign has the following total energy, with $E(k)$ denoting the complete elliptic integral of the second kind:

$$\begin{aligned} E[\psi] &= \int_0^L d\tau \left(\frac{1}{2}(\partial_\tau \psi)^2 + V(1, \psi) \right) \\ &= \frac{\sqrt{2} \cdot 4nm^2(m+1)}{3(1+k^2)^{\frac{3}{2}}} \\ &\quad \left(\frac{1}{4m^2} (3k^4 + (6 + 32m^2 + 24m)k^2 + 16m^2 - 24m + 3) K(k) \right. \\ &\quad \left. + 2 \left(\frac{3}{m} - 2 \right) (1+k^2) E(k) \right). \end{aligned} \quad (4.11)$$

4.3 Dyons

Replacing the coordinate τ in \mathbb{R} direction by $i\tau$ changes the signature of the metric from Riemannian to Lorentzian:

$$g = -e^0 e^0 + e^1 e^1 + e^{2h} \delta_{ab} e^{ab}. \quad (4.12)$$

The Yang-Mills equations (3.21) remain unchanged, except for the fact that the second-order derivatives now come with a minus sign:

$$(\ddot{\chi}, \ddot{\psi}) \rightarrow (-\ddot{\chi}, -\ddot{\psi}). \quad (4.13)$$

This corresponds to a sign flip of the potential, so that we have to study V instead of $-V$. Dyons are finite-energy solutions to the second-order equations obtained by this sign flip. Just as Yang-Mills solutions, they can interpolate between two critical points (kink), or start and end at the same point (bounce). Solutions that oscillate around a minimum

can exist as well, but they do not lead to finite energy and hence will not be considered in the following.

4.4 Discussion and summary of solutions

Recall that in our sign convention, instanton solutions interpolate between minima and dyon solutions between maxima of V . In both cases, solutions that start or end at a saddle point are possible as well. With this in mind, we can expect the following solutions:

- κ arbitrary: there exist at least two zero-potential critical points at $(0, \pm\sqrt{2m})$ for all κ . According to Appendix C, they can be minima or saddle points of V , depending on the value of κ . This means that we can always find interpolating solutions, either of dyon or of Yang-Mills type. These solutions have to be constructed numerically unless $\kappa = \frac{1}{1-m}$.
- $\kappa = 1$: this is the instanton case. Yang-Mills solutions exist between $(0,0)$ and $(1, \pm\sqrt{2m})$ (cf. [20]). We do not expect to find any finite-action dyon solutions, as the zero-potential critical points of V are minima.
- $\kappa = \frac{1}{1-m}$ ($\kappa = -1$ for $m = 2$): in this case, we find three nonzero critical points along the χ axis. An analytic Yang-Mills solution interpolates between the two remaining zero-potential critical points, which are minima for all m . This solution for arbitrary m is presented in Chapter 4.1.
- $\kappa = \frac{3}{1-m}$ ($\kappa = -3$ for $m = 2$): we find four zero-potential critical points. Two of them are located at the lines with $\chi = 1$ and $\chi = -1$, respectively. We do not find any analytic solutions along the $\chi = \pm 1$ and $\chi = 0$ axes. There should, however, be a number of numerical solutions interpolating between various pairs of critical points.

We do not expect any analytic dyon solutions, as the zero-potential critical points are minima in the analytically solvable cases. For a better understanding, we present the case $m = 2$ as an example. The potential for various interesting values of κ is shown in Figure 1, and further dyon and Yang-Mills solutions for this example are presented in Figures 2 and 4. The list of zero-potential critical points can be found in Table 3.

	κ	Eigenvalues of Jacobian
$(\chi_1, \psi_1) = (0, 0)$	1	$9, \frac{9}{4}$
$(\chi_2, \psi_2) = (1, \pm 2)$	any	$\frac{9}{4}(5 + \kappa + \sqrt{5}(1 + \kappa)), \frac{9}{4}(5 + \kappa - \sqrt{5}(1 + \kappa))$
$(\chi_3, \psi_3) = (1, -2)$	$-\sqrt{5}$	$\frac{9}{2}(5 - \sqrt{5}), 0$
$(\chi_4, \psi_4) = (-1, \pm 2)$	-3	$\frac{9}{2}(1 + \sqrt{5}), \frac{9}{2}(1 - \sqrt{5})$
$(1, 2)$	$\sqrt{5}$	$\frac{9}{2}(5 + \sqrt{5}), 0$

Table 3: Critical points and corresponding κ values with vanishing potential for $m = 2$

5 Conclusion and outlook

Using a special ansatz for the gauge connection, we have derived a system of explicit second-order Yang-Mills equations on the cylinder over a class of Sasakian manifolds. We have constructed the corresponding action and potential, discussed the behaviour of the critical zero-potential points and found analytic as well as numerical solutions of Yang-Mills, dyon and sphaleron type.

A similar discussion for cylinders over certain $SU(3)$ -structure manifolds can be found in [22]. A comparison with our results illustrates that Sasakian and $SU(3)$ -structures are fundamentally different. The perhaps most striking fact is that the 3-symmetry of the $SU(3)$ -structure manifold is recovered in the shape of the potential, whereas the potential in the Sasakian case is symmetric only under sign changes of the variable ψ . Furthermore, the Sasakian potential does not admit as many solutions with straight trajectories in the (χ, ψ) -plane as the $SU(3)$ -structure potential does. In the latter case, the distribution of κ -dependent and κ -independent zero-potential critical points allows to systematically associate certain types of solutions (kinks, bounces) to intervals of the deformation parameter κ . In particular, there are always three critical points on the real axis. The Sasakian potential admits fewer κ -independent zero-potential critical points, and they are not as regularly distributed as in the $SU(3)$ -structure case. The range and type of our solutions is therefore significantly different.

In spite of these differences, we have found that Sasakian manifolds do admit various interesting solutions. This, and in particular the fact that we have found an analytic kink-type solution of the Yang-Mills equation, makes them potentially interesting for

non-supersymmetric string compactifications. It may be worth studying the instanton solution (4.7) in the context of the AdS/CFT duality mentioned in the introduction.

To complete the discussion, it would be interesting to consider Yang-Mills and dyon solutions on cylinders over G_2 -structure manifolds, i. e. 8-dimensional manifolds with $Spin(7)$ -structure. We are planning to present results for this case in the near future. In addition, the analysis for the 3-Sasakian case is still missing. Another open question is how the Yang-Mills and dyon solutions change when considering cones and sine-cones instead of cylinders. On conical manifolds, the second-order equations acquire a first-order friction term, hence the analysis might have to be done numerically.

A Sum of structure constants

Proposition 1. *Let M be a Sasakian manifold of dimension $2m + 1$ with structure group $SU(m)$ and metric*

$$g_M = e^1 e^1 + \frac{2m}{m+1} \delta_{ab} e^{ab}. \quad (\text{A.1})$$

Let $\mathbb{R} \times M$ be the cylinder with structure group $SU(m+1)$. Then the Lie algebras corresponding to the structure groups admit a splitting $\mathfrak{su}(m+1) = \mathfrak{su}(m) \oplus \mathfrak{m}$, as described in Chapter 3. We use indices $a = (1, 2, \dots, \dim \mathfrak{m})$ to label the generators of \mathfrak{m} and indices i, j for the remaining generators of $\mathfrak{su}(m)$. In this setup, the $SU(m+1)$ -structure constants satisfy equation (3.20):

$$f_{ac}^i f_{ib}^c = \frac{2(m^2 - 1)}{m} \delta_{ab}. \quad (\text{A.2})$$

Proof: To see this, note first that the components of the metric (A.1) take the form

$$(g_M)_{11} = 1, \quad (g_M)_{ab} = \frac{2m}{m+1} \delta_{ab}. \quad (\text{A.3})$$

The Killing form of $\mathfrak{su}(m+1)$ induces the following metric on \mathfrak{m} :

$$(g_K)_{\mu\nu} = f_{\mu\tilde{c}}^{\tilde{d}} f_{\tilde{d}\nu}^{\tilde{c}}. \quad (\text{A.4})$$

The structure constants are normalized such that they satisfy

$$f_{ab}^1 = 2P_{ab1}, \quad f_{1a}^b = \frac{m+1}{m} P_{1ab}. \quad (\text{A.5})$$

Hence, the Killing metric takes the following values:

$$(g_K)_{11} = f_{1\tilde{c}}^{\tilde{d}} f_{\tilde{d}1}^{\tilde{c}} = f_{1c}^d f_{d1}^c = \frac{2(m+1)^2}{m} =: X, \quad (\text{A.6})$$

$$(g_K)_{ab} = 2(f_{a1}^d f_{db}^1 + f_{ai}^d f_{db}^i) = 2 \left(\frac{2(m+1)}{m} \delta_{ab} + f_{ai}^d f_{db}^i \right). \quad (\text{A.7})$$

This metric matches the metric (A.1) up to rescaling of structure constants by the factor \sqrt{X} . We therefore find

$$(g_M)_{11} = \frac{1}{X} (g_K)_{11} = 1, \quad (\text{A.8})$$

$$\begin{aligned} (g_M)_{ab} &= \frac{1}{X} (g_K)_{ab} \\ &= \frac{2}{X} (f_{a1}^d f_{db}^1 + f_{ai}^d f_{db}^i) \\ &= \frac{2}{X} \left(\frac{2(m+1)}{m} \delta_{ab} + f_{ai}^d f_{db}^i \right) \\ &= \frac{2m}{m+1} \delta_{ab}. \end{aligned} \quad (\text{A.9})$$

We conclude that both summands in $(g_M)_{ab}$ must be proportional to δ_{ab} , hence $f_{ai}^d f_{db}^i \stackrel{!}{=} \beta \delta_{ab}$ with some real parameter $\beta \in \mathbb{R}$. This leads to

$$\frac{2}{X} \left(\frac{2(m+1)}{m} + \beta \right) \delta_{ab} = \frac{2m}{m+1} \delta_{ab} \quad \Rightarrow \quad \beta = \frac{2(m^2 - 1)}{m} \quad (\text{A.10})$$

and proves equation (A.2). \square

B Action

Proposition 2. *The Yang-Mills equations (3.21) on the cylinder over a Sasakian manifold are equations of motion for the action*

$$\begin{aligned} S &= \frac{m}{4(m+1)} \int_{\mathbb{R} \times M} \text{tr} \left(\mathcal{F} \wedge * \mathcal{F} + 2 \left(\frac{m}{m+1} \right)^2 \kappa d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F} \right) \\ &= \text{Vol}(M) \times \int_{\mathbb{R}} \left[-\frac{1}{2} (\dot{\chi}^2 + \dot{\psi}^2) - \left(\frac{m+1}{m} \right)^2 \right. \\ &\quad \left(\psi^2 (1 - \chi)^2 + m(1 - m)(1 - \kappa) \left(\frac{1}{2m} \psi^2 - 1 \right)^2 \right. \\ &\quad \left. \left. + m(1 + \kappa(m - 1)) \left(\chi - \frac{1}{2m} \psi^2 \right)^2 \right) \right] d\tau \end{aligned} \quad (\text{B.1})$$

with potential

$$\begin{aligned} V(\chi, \psi) &= \frac{1}{2} \left(\frac{m+1}{m} \right)^2 \left(((1 + \kappa(m - 1))m\chi^2 + (\kappa(1 - m) - 3)\chi\psi^2 \right. \\ &\quad \left. + \chi^2\psi^2 + (2 - m + \kappa(m - 1))\psi^2 + \frac{1}{4}\psi^4 + m(m - 1)(1 - \kappa) \right), \end{aligned} \quad (\text{B.2})$$

where $*_M$ denotes the Hodge star operator on the Sasakian manifold M with respect to the metric g_M , and $*$ denotes the Hodge star operator on the cylinder.

Proof: To see this, we compute the summands $\text{tr}(\mathcal{F} \wedge * \mathcal{F})$ and $\text{tr}(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F})$ separately. For the first summand, we find

$$\begin{aligned}
\text{tr}(\mathcal{F} \wedge * \mathcal{F}) &= \frac{1}{2} \text{tr} (2\mathcal{F}_{0\mu} \mathcal{F}^{0\mu} + \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) \text{Vol}(\mathbb{R} \times M) \\
&= \frac{1}{2} \text{tr} \left(2\mathcal{F}_{01} \mathcal{F}_{01} + \frac{m+1}{m} \mathcal{F}_{0a} \mathcal{F}_{0a} \right. \\
&\quad \left. + \frac{m+1}{m} \mathcal{F}_{1b} \mathcal{F}_{1b} + \left(\frac{m+1}{2m} \right)^2 \mathcal{F}_{ab} \mathcal{F}_{ab} \right) \text{Vol}(\mathbb{R} \times M) \\
&= \frac{1}{2} \left(-4 \frac{m+1}{m} (\dot{\chi}^2 + \dot{\psi}^2) - 4 \left(\frac{m+1}{m} \right)^3 \psi^2 (1 - \chi)^2 \right. \\
&\quad \left. + \left(\frac{m+1}{2m} \right)^2 \left(\left(\frac{1}{2m} \psi^2 - 1 \right)^2 f_{ab}^i f_{ab}^j f_{im}^n f_{jn}^m \right. \right. \\
&\quad \left. \left. - 16(m+1) \left(\chi - \frac{1}{2m} \psi^2 \right)^2 \right) \right) \text{Vol}(\mathbb{R} \times M), \tag{B.3}
\end{aligned}$$

using $\text{Vol}(\mathbb{R} \times M) = \sqrt{|g_Z|} d\tau \wedge e^{1 \cdots (2m+1)}$, the components (3.19) of the curvature and the following explicit expressions for the trace of I_i, I_μ in the representation (2.21):

$$\text{tr}(I_1 I_1) = I_{10}^a I_{1a}^0 + I_{1a}^b I_{1b}^a + I_{11}^0 I_{10}^1 + I_{10}^1 I_{11}^0 = -2 \frac{m+1}{m}, \tag{B.4}$$

$$\text{tr}(I_i I_j) = I_{ia}^b I_{jb}^a, \tag{B.5}$$

$$\text{tr}(I_1 I_j) = 0, \tag{B.6}$$

$$\text{tr}(I_a I_a) = 2(I_{a0}^b I_{ab}^0 + I_{a1}^b I_{ab}^1) = -8m \text{ (sum over } a). \tag{B.7}$$

The combination $f_{ab}^i f_{ab}^j f_{im}^n f_{jn}^m$ of structure constants in equation (B.3) can be simplified by use of the following relation. The commutator of two generators in the representation (2.21) takes the form

$$[I_a, I_b]_c^d = f_{ab}^i I_{ic}^d + f_{ab}^1 I_{1c}^d. \tag{B.8}$$

Inserting the explicit expressions for I_1 and I_a leads to the identity

$$f_{ab}^i f_{ic}^d = \omega_{bc} \omega_{ad} - \omega_{ac} \omega_{bd} - \delta_a^c \delta_b^d + \delta_b^c \delta_a^d + \frac{2}{m} P_{ab1} \omega_{cd}. \tag{B.9}$$

We use this expression to rewrite the sum of structure constants in equation (B.3) and

find

$$\begin{aligned}
\sum_{a,b,c,d,i,j} f_{ab}^i f_{ab}^j f_{im}^n f_{jn}^m &= \sum_{a,b,c,d,i,j} \left(\omega_{bc}\omega_{ad} - \omega_{ac}\omega_{bd} - \delta_a^c \delta_b^d + \delta_b^c \delta_a^d + \frac{2}{m} \omega_{ab}\omega_{cd} \right) \\
&\quad \left(\omega_{bd}\omega_{ac} - \omega_{ad}\omega_{bc} - \delta_a^d \delta_b^c + \delta_b^d \delta_a^c - \frac{2}{m} \omega_{ab}\omega_{cd} \right) \\
&= \sum_{a,b,c,d,i,j} \left(2\omega_{bc}\omega_{ad}\omega_{bd}\omega_{ac} - 2\omega_{ac}\omega_{bd}\omega_{ac}\omega_{bd} + \frac{4}{m} \omega_{ab}\omega_{cd}\omega_{bd}\omega_{ac} \right. \\
&\quad \left. - \frac{4}{m} \omega_{ab}\omega_{cd}\omega_{bc}\omega_{ad} - \frac{4}{m^2} \omega_{ab}\omega_{cd}\omega_{ab}\omega_{cd} \right. \\
&\quad \left. + \left(\frac{8}{m} - 4 \right) \omega_{ab}\omega_{ab} + 2(\delta_a^d \delta_a^d - \delta_a^d \delta_b^c) \right) \\
&= 4m - 8m^2 + 8 + 8 - 16 + \left(\frac{8}{m} - 4 \right) 2m + 4m - 8m^2 \\
&= 16(1 - m^2), \tag{B.10}
\end{aligned}$$

using the fact that $\omega_{ab}\omega_{ab} = 2m$ and that only the components of ω_{ab} with $b = a + 1$ or $b = a - 1$ are nonzero. To avoid confusion, the summation indices have been explicitly displayed at this point. Note that all indices are being summed over. Inserting this back into equation (B.3) yields

$$\begin{aligned}
\text{tr}(\mathcal{F} \wedge * \mathcal{F}) &= 4 \frac{m+1}{m} \left(-\frac{1}{2} (\dot{\chi}^2 + \dot{\psi}^2) - \left(\frac{m+1}{m} \right)^2 \psi^2 (1 - \chi)^2 \right. \\
&\quad \left. + (1 - m^2) \frac{m+1}{m} \left(\frac{1}{2m} \psi^2 - 1 \right)^2 \right. \\
&\quad \left. - \frac{(m+1)^2}{m} \left(\chi - \frac{1}{2m} \psi^2 \right)^2 \right) \text{Vol}(\mathbb{R} \times M) \\
&= 4 \frac{m+1}{m} \left(-\frac{1}{2} (\dot{\chi}^2 + \dot{\psi}^2) - \left(\frac{m+1}{m} \right)^2 (\psi^2 (1 - \chi)^2 \right. \right. \\
&\quad \left. \left. - (1 - m)m \left(\frac{1}{2m} \psi^2 - 1 \right)^2 \right. \right. \\
&\quad \left. \left. + m \left(\chi - \frac{1}{2m} \psi^2 \right)^2 \right) \right) \text{Vol}(\mathbb{R} \times M). \tag{B.11}
\end{aligned}$$

For the second summand in the action, note that $d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}$ is a form of top degree in on the cylinder $\mathcal{Z}(M)$. A convenient way to compute the components of this

form is to apply the Hodge star operator. We find

$$\begin{aligned}
*_{{\mathcal{Z}}(M)}(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) &= *_M(*_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) \\
&= \frac{1}{4n!(n-4)!} Q_{\mu\nu\rho\sigma} \mathcal{F}^{\alpha\beta} \mathcal{F}^{\gamma\delta} \epsilon^{\mu\nu\rho\sigma\xi_1\cdots\xi_{n-4}} \epsilon_{\xi_1\cdots\xi_{n-4}\alpha\beta\gamma\delta} \\
&= \frac{1}{4} Q_{\mu\nu\rho\sigma} \mathcal{F}^{\alpha\beta} \mathcal{F}^{\gamma\delta} \delta_{\alpha\beta\gamma\delta}^{\mu\nu\rho\sigma} \\
&= \frac{1}{4} Q_{\mu\nu\rho\sigma} \mathcal{F}^{[\mu\nu} \mathcal{F}^{\rho\sigma]} \\
&= 3\omega_{\mu\nu}\omega_{\rho\sigma} \mathcal{F}^{[\mu\nu} \mathcal{F}^{\rho\sigma]}, \tag{B.12}
\end{aligned}$$

using $n := 2m + 1 = \dim M$ as well as $Q = \frac{1}{2}\omega \wedge \omega \Leftrightarrow Q_{\mu\nu\rho\sigma} = 4!\omega_{\mu\nu}\omega_{\rho\sigma}$. This result implies

$$d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F} = 3\omega_{ab}\omega_{cd} \mathcal{F}^{[ab} \mathcal{F}^{c]d} Vol(\mathbb{R} \times M). \tag{B.13}$$

We find

$$\begin{aligned}
&\text{tr}(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) \\
&= 3\omega_{ab}\omega_{cd} \text{tr}(\mathcal{F}^{[ab} \mathcal{F}^{c]d}) Vol(\mathbb{R} \times M) \\
&= \left(\frac{m+1}{2m}\right)^4 \left(\left(\frac{1}{2m}\psi^2 - 1\right)^2 3\omega_{ab}\omega_{cd} f_{[ab}^i f_{c]d}^j f_{im}^n f_{jn}^m \right. \\
&\quad \left. - 8\frac{m+1}{m} \left(\chi - \frac{1}{2m}\psi^2\right)^2 3\omega_{ab}\omega_{cd} P_{[ab|1|} P_{c]d1} \right) Vol(\mathbb{R} \times M) \\
&= \left(\frac{m+1}{2m}\right)^4 \left(2\left(\frac{1}{2m}\psi^2 - 1\right)^2 \omega_{ab}\omega_{cd} f_{bc}^i f_{ad}^j f_{im}^n f_{jn}^m \right. \\
&\quad \left. - 32(m^2 - 1) \left(\chi - \frac{1}{2m}\psi^2\right)^2 \right) Vol(\mathbb{R} \times M), \tag{B.14}
\end{aligned}$$

by use of $f_{ab}^1 f_{ab}^i = 0$ and $3\omega_{ab}\omega_{cd} P_{[ab|1|} P_{c]d1} = 4m(m-1)$. The sum of structure constants simplifies to

$$\omega_{ab}\omega_{cd} f_{[ab}^i f_{c]d}^j f_{im}^n f_{jn}^m = 16(m^2 - 1). \tag{B.15}$$

This identity is proven by writing the structure constants in terms of equation (B.9) and evaluating all sums explicitly. As the computation follows the same pattern as the derivation of equation (B.10), we do not present the details here. We find

$$\begin{aligned}
&\text{tr}(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) \\
&= \left(\frac{m+1}{2m}\right)^4 32(m^2 - 1) \left(\left(\frac{1}{2m}\psi^2 - 1\right)^2 - \left(\chi - \frac{1}{2m}\psi^2\right)^2 \right) Vol(\mathbb{R} \times M). \tag{B.16}
\end{aligned}$$

Now the identities (B.11) and (B.16) can be inserted into the action. This leads to the result (B.1), taking into account that the volume form on the cylinder satisfies $Vol(\mathbb{R} \times M) = d\tau \wedge Vol(M)$. \square

C Eigenvalues of the Hesse matrix

Let us once again consider the potential (4.2):

$$V(\chi, \psi) = \frac{1}{2} \left(\frac{m+1}{m} \right)^2 \left(((1 + \kappa(m-1))m\chi^2 + (\kappa(1-m) - 3)\chi\psi^2 + \chi^2\psi^2 + (2-m + \kappa(m-1))\psi^2 + \frac{1}{4}\psi^4 + m(m-1)(1-\kappa)) \right). \quad (C.1)$$

The critical points (χ, ψ) of V that satisfy $\partial_\chi V = \partial_\psi V = 0$ are listed in equation (4.4), and the eigenvalues of the matrix

$$\begin{pmatrix} \frac{\partial^2 V}{\partial \chi^2} & \frac{\partial^2 V}{\partial \chi \partial \psi} \\ \frac{\partial^2 V}{\partial \psi \partial \chi} & \frac{\partial^2 V}{\partial \psi^2} \end{pmatrix} \quad (C.2)$$

have been presented in Table 1. The eigenvalues at the critical point $(\chi_2, \psi_2) = (1, \pm\sqrt{2m})$ need a more detailed discussion. They are given by

$$(\lambda_1, \lambda_2) = \left(\frac{1}{2} \left(\frac{m+1}{m} \right)^2 \left((5 + \kappa(m+1))m + (1 + \kappa(m-1))\sqrt{m(8+m)} \right), \right. \\ \left. \frac{1}{2} \left(\frac{m+1}{m} \right)^2 \left((5 + \kappa(m-1))m - (1 + \kappa(m-1))\sqrt{m(8+m)} \right) \right). \quad (C.3)$$

λ_1 is greater than zero for

$$\kappa > \kappa_+ := -\frac{5m + \sqrt{m(8+m)}}{m(m+1) + (m-1)\sqrt{m(8+m)}} \quad (C.4)$$

and smaller than zero otherwise. λ_2 is greater than zero for

$$\kappa < \kappa_- := \frac{-5m + \sqrt{m(8+m)}}{m(m-1) - (m-1)\sqrt{m(8+m)}} \quad (C.5)$$

and smaller otherwise. We have $\kappa_- > \kappa_+$ for any positive integer value of $m > 1$. The extremum of the potential at $(1, \pm\sqrt{2m})$ is therefore

$$\begin{array}{llll} 1) & \text{a saddle} & \text{for} & \kappa > \kappa_-, \\ 2) & \text{indefinite} & \text{for} & \kappa = \kappa_-, \\ 3) & \text{a minimum} & \text{for} & \kappa_- > \kappa > \kappa_+, \\ 4) & \text{indefinite} & \text{for} & \kappa = \kappa_+, \\ 5) & \text{a saddle} & \text{for} & \kappa_+ > \kappa. \end{array} \quad (C.6)$$

This observation is in agreement with the remaining cases listed in Table 1: since $\kappa_+ > \frac{3}{1-m}$, we find one positive and one negative eigenvalue for (χ_4, ψ_4) .

We can expect Yang-Mills solutions when the extrema at $(1, \pm\sqrt{2m})$ are minima, i. e. for $\kappa_- > \kappa > \kappa_+$ (in particular for $\kappa = 1$), or saddle points, and dyon solutions when they are saddle points. As λ_1 and λ_2 do not simultaneously become smaller than zero for any fixed value of κ , the critical points never become maxima.

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References

- [1] M. Green, J. H. Schwarz and E. Witten, “Superstring Theory,” Volume 1, Cambridge University Press (1987).
- [2] D. Lüst and R. Blumenhagen, “Basic Concepts of String Theory,” **Springer Verlag Berlin** (1987).
- [3] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, “First-Order Equations for Gauge Fields in Spaces of Dimension Greater than Four,” **Nucl. Phys. B** **214** (1983) 452.
- [4] D. B. Fairlie and J. Nuyts, “Spherically Symmetric Solutions of Gauge Theories in Eight Dimensions,” **J. Phys. A** **17** (1984) 2867.
- [5] S. Fubini and H. Nicolai, “The Octonionic Instanton,” **Phys. Lett. B** **155** (1985) 369.
- [6] T. A. Ivanova and O. Lechtenfeld, “Yang-Mills Instantons and Dyons on Group Manifolds,” **Phys. Lett. B** **670** (2008) 91, [arXiv:0806.0394](#).
- [7] T. A. Ivanova, O. Lechtenfeld, A. D. Popov and T. Rahn, “Instantons and Yang-Mills Flows on Coset Spaces,” **Lett. Math. Phys.** **89** (2009) 231, [arXiv:0904.0654](#).
- [8] D. Harland, T. A. Ivanova, O. Lechtenfeld and A. D. Popov, “Yang-Mills Flows on Nearly Kahler Manifolds and G_2 -Instantons,” **Commun. Math. Phys.** **300** (2010) 185, [arXiv:0909.2730](#).
- [9] A. S. Haupt, T. A. Ivanova, O. Lechtenfeld and A. D. Popov, “Chern-Simons Flows on Aloff-Wallach Spaces and Spin(7)-Instantons,” **Phys. Rev. D** **83** (2011) 105028, [arXiv:1104.5231](#).

- [10] K. P. Gemmer, O. Lechtenfeld, C. Nolle and A. D. Popov, “Yang-Mills Instantons on Cones and Sine-Cones over Nearly Kahler Manifolds,” *JHEP* **1109** (2011) 103, [arXiv:1108.3951](#).
- [11] M. Grana, “Flux Compactifications in String Theory: A Comprehensive Review,” *Phys. Rept.* **423** (2006) 910, [arXiv:hep-th/0509003](#).
- [12] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, “Four-Dimensional String Compactifications with D-Branes, Orientifolds and Fluxes,” *Phys. Rept.* **445** (2007) 1, [arXiv:hep-th/0610327](#).
- [13] M. R. Douglas and S. Kachru, “Flux Compactification,” *Rev. Mod. Phys.* **79** (2007) 733, [arXiv:hep-th/0610102](#).
- [14] C. Bär, “Real Killing Spinors and Holonomy,” *Communications in Mathematical Physics*, June (II) 1993, Volume 154, Issue 3, pp 509-521.
- [15] D. Cassani, G. Dall’Agata and A. F. Faedo, “Type IIB Supergravity on Squashed Sasaki-Einstein Manifolds,” *JHEP* **1005** (2010) 094, [arXiv:1003.4283](#).
- [16] J. P. Gauntlett and O. Varela, “Universal Kaluza-Klein Reductions of Type IIB to $N = 4$ Supergravity in Five Dimensions,” *JHEP* **1006** (2010) 081, [arXiv:1003.5642](#).
- [17] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, “A New Infinite Class of Sasaki-Einstein Manifolds,” *Adv. Theor. Math. Phys.* **8** (2006) 987, [arXiv:hep-th/0403038](#).
- [18] F. P. Correia, “Hermitian Yang-Mills Instantons on Calabi-Yau Cones,” *JHEP* **0912** (2009) 004, [arXiv:0910.1096](#).
- [19] F. P. Correia, “Hermitian Yang-Mills Instantons on Resolutions of Calabi-Yau Cones,” *JHEP* **1102** (2011) 054, [arXiv:1009.0526](#).
- [20] D. Harland and C. Nölle, “Instantons and Killing Spinors,” *JHEP* **1203** (2012) 082, [arXiv:1109.3552](#).
- [21] T. Rahn, “Yang-Mills Configurations on Coset Spaces,” Master’s thesis, Leibniz Universität Hannover (2009).
- [22] I. Bauer, T. A. Ivanova, O. Lechtenfeld and F. Lubbe, “Yang-Mills Instantons and Dyons on Homogeneous G_2 -Manifolds,” *JHEP* **1010** (2010) 044, [arXiv:1006.2388](#).
- [23] T. A. Ivanova and A. D. Popov, “Instantons on Special Holonomy Manifolds,” *Phys. Rev. D* **85** (2012) 105012, [arXiv:arXiv:1203.2657](#).
- [24] J. Sparks, “Sasaki-Einstein Manifolds,” *Surveys Diff. Geom.* **16** (2011) 265, [arXiv:arXiv:1004.2461](#).
- [25] C. P. Boyer and K. Galicki, “Sasakian Geometry,” Oxford University Press (2008).
- [26] A. Fino, L. Vezzoni and A. Andrada, “A Class of Sasakian 5-Manifolds,” [arXiv:0807.1800](#).

- [27] D. Conti and A. Fino, “Calabi-Yau Cones from Contact Reduction,”
[arXiv:0710.4441](#).
- [28] M. Tormählen, “Yang-Mills Solutions on Manifolds with G-Structure,” PhD thesis,
Leibniz Universität Hannover (2015).
- [29] K. Becker, M. Becker and J. H. Schwarz, “String Theory and M-Theory,”
Cambridge University Press (2007).
- [30] N. S. Manton and P. Sutcliffe, “Topological Solitons,” Cambridge, UK: Univ. Pr.
(2004) 493 p.
- [31] W. Schwalm, “Elliptic Functions sn, cn, dn , as Trigonometry,”
http://www.und.edu/instruct/schwalm/MAA_Presentation_10-02/handout.pdf.
- [32] N. S. Manton and T. M. Samols, “Sphalerons On A Circle,” **Phys. Lett. B** **207**
(1988) 179.

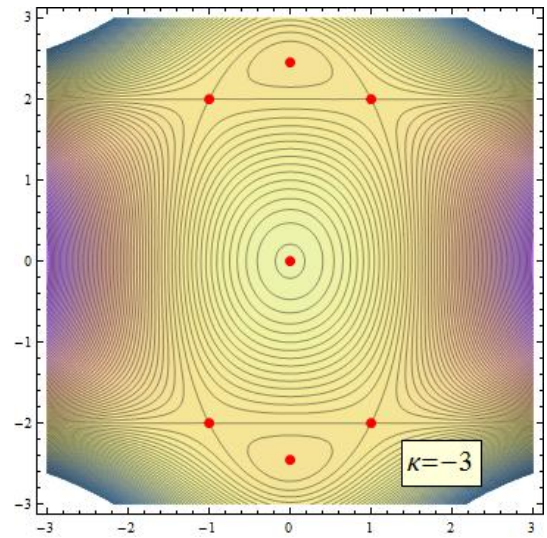
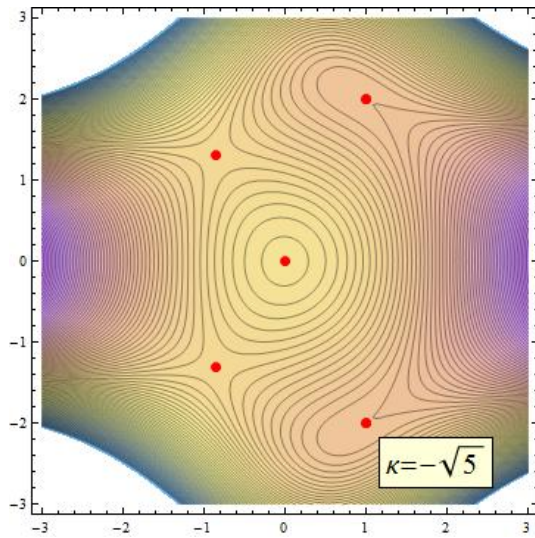
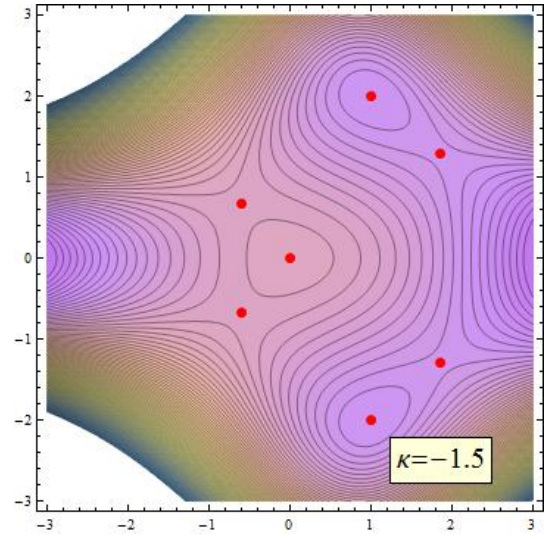
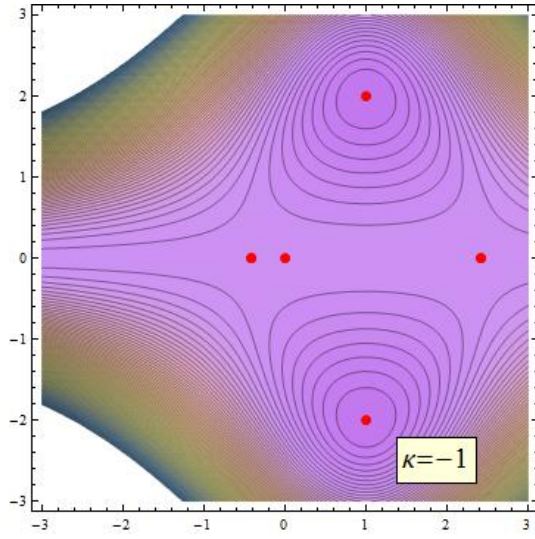
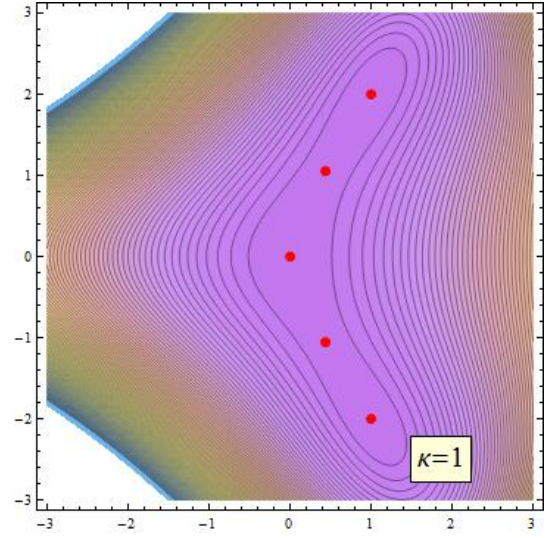
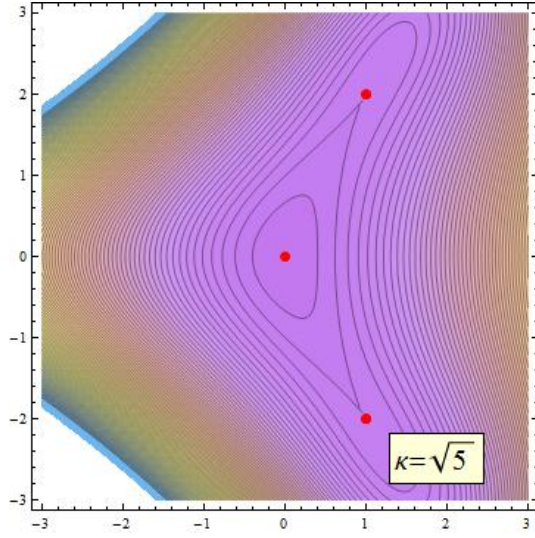


Figure 1: Potential plots for various values of κ and $m = 2$

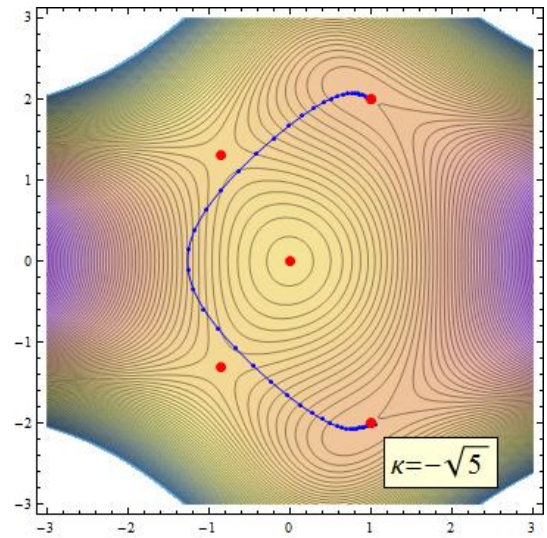
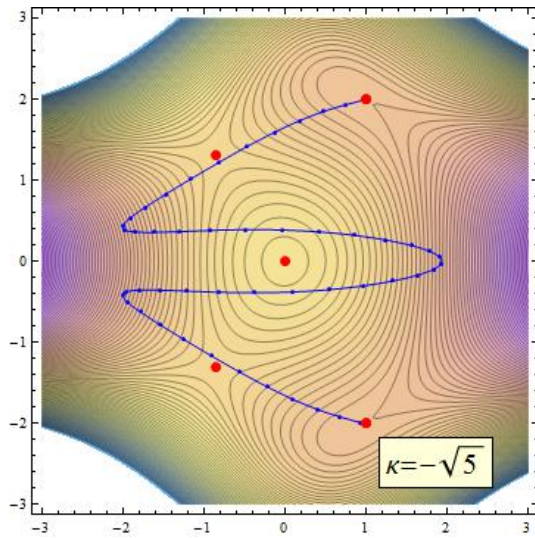
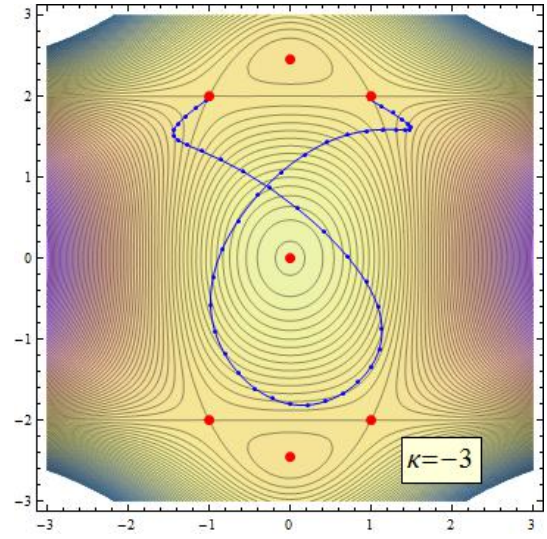
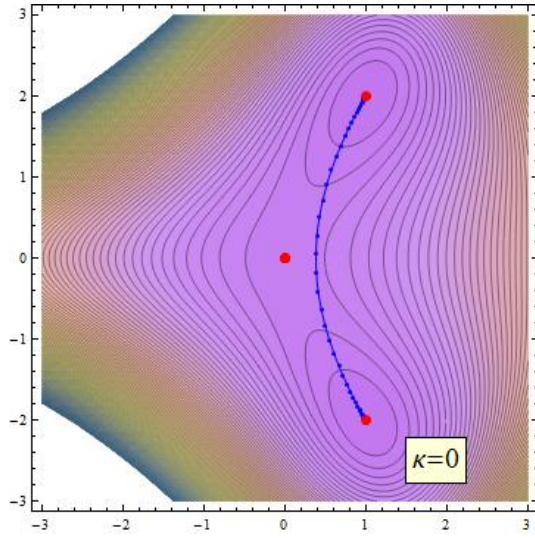
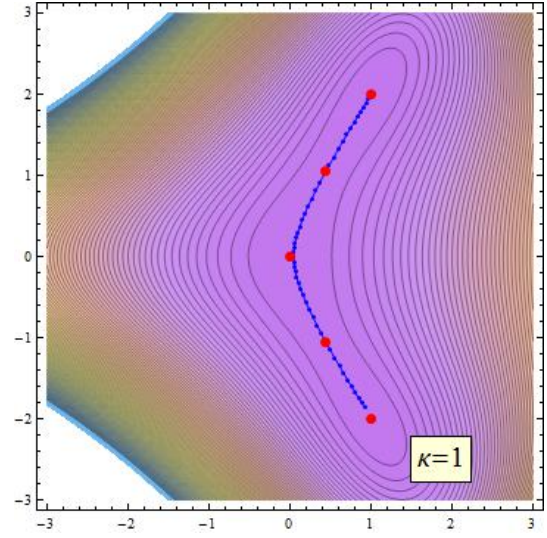
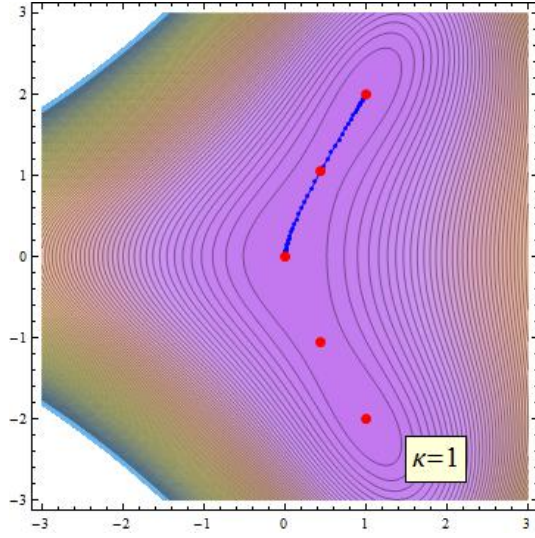


Figure 2: Some solutions of the Yang-Mills equation for various values of κ and $m = 2$

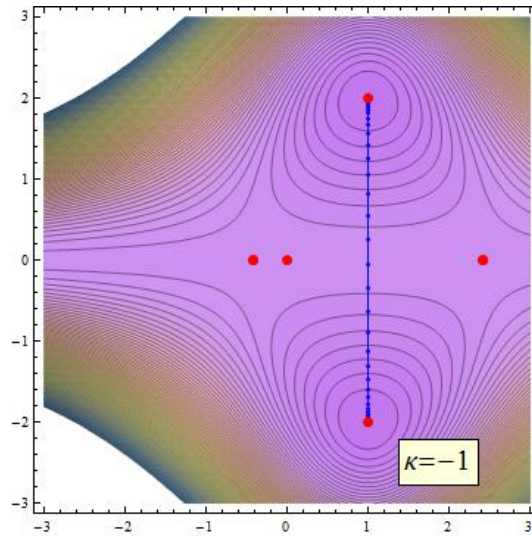


Figure 3: Analytic solution of the Yang-Mills equation for $\kappa = -1$ and $m = 2$

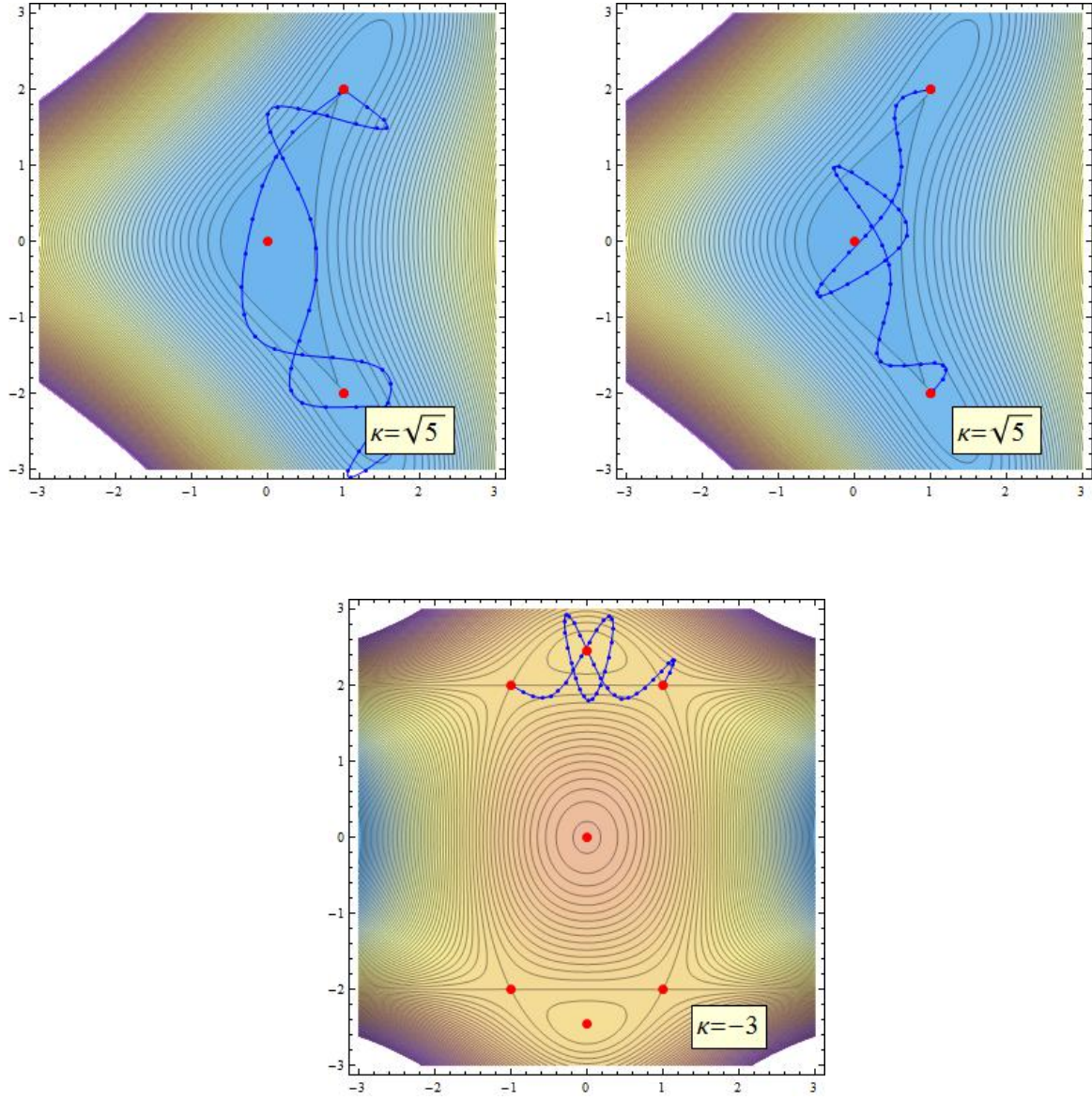


Figure 4: Some numerical dyon solutions for various values of κ and $m = 2$